Directed Univalence in Bicubical Directed Type Theory

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Dan Licata and Matthew Weaver

Wesleyan Princeton

Overview of Directed Univalence

- **Step 0.** Construct function $\mathrm{dua}^{-1} : \mathrm{Hom}_{\mathrm{UCov}}\ A\ B \to \mathrm{El}\ A \to \mathrm{El}\ B$
- **Step 1.** Define $\mathrm{dua} : (A\ B : U)\ (f : A \to B) \to \mathrm{Hom}_{U}\ A\ B$
- **Step 2.** Show dua is fibrant (Kan + Covariant)
  - $\mathrm{dua} : (A\ B : \mathrm{UCov})\ (f : \mathrm{El}\ A \to \mathrm{El}\ B) \to \mathrm{Hom}_{\mathrm{UCov}}\ A\ B$
- **Step 3.** Prove dua and dua^{-1} are inverses
- We have completed/formalized steps 0-2 and that $\mathrm{El}\ A \to \mathrm{El}\ B$ is a retract of $\mathrm{Hom}_{\mathrm{UCov}}\ A\ B$

Glue Types

$\mathrm{Glue}\ [\ \alpha \mapsto (T\ ,\ f)\ ]\ B :=$

$$
\begin{tikzcd}
\alpha \vdash T \arrow[d, "\alpha \vdash f"'] \\
B
\end{tikzcd}
$$

$\alpha \vdash \mathrm{Glue}\ [\ \alpha \mapsto (T\ ,\ f)\ ]\ B \equiv T$

$\mathrm{glue}\ t\ b :=$

$$
\begin{tikzcd}
\alpha \vdash t : T \arrow[d, "\alpha \vdash f"'] \\
b : B
\end{tikzcd}
$$

$$\frac{g : \mathrm{Glue}\ [\ \alpha \mapsto (T\ ,\ f)\ ]\ B}{\mathrm{unglue}\ g : B}$$

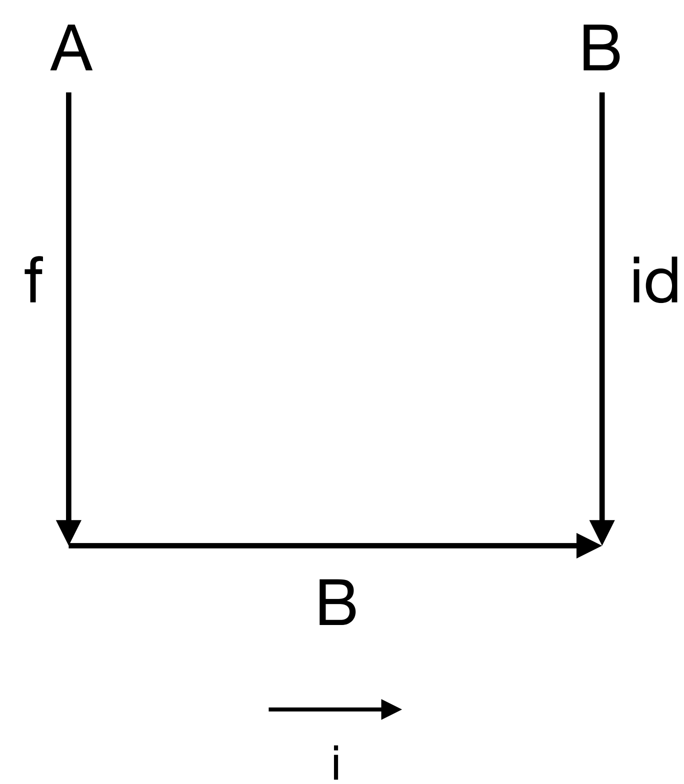
$\alpha \vdash \mathrm{glue}\ t\ b \equiv t$

$\alpha \vdash \mathrm{unglue}\ (\mathrm{glue}\ t\ b) \equiv f\ t$

$\mathrm{glue}\ g\ (\mathrm{unglue}\ g) \equiv g$

Directed Univalence

$$\begin{aligned}\text{dua } i\ A\ B\ f := \text{Glue } [\ & i = 0 \mapsto (A\ ,\ f : A \to B) \\ ,\ & i = 1 \mapsto (B\ ,\ \text{id})\ ]\ B \qquad : \text{Hom}_U\ A\ B\end{aligned}$$



(Same construction used to get regular univalence in cubical type theory)

dua is Covariant

$$\frac{\Gamma : U \quad i : \Gamma \to \mathbb{2} \quad A : \Gamma \to \mathrm{UCov} \quad B : \Gamma \to \mathrm{UCov} \\ f : (x : \Gamma) \to A\ x \to B\ x \quad p : \mathbb{2} \to \Gamma \quad \alpha : \mathrm{Cofib} \\ \alpha \vdash t : (j : \mathbb{2}) \to \mathrm{dua}\ i\ A\ B\ f\ (p\ j) \quad b : \mathrm{duaF}\ i\ A\ B\ f\ (p\ 0)\ [\ \alpha \mapsto t\ 0\ ]}{\mathrm{duaF}\ i\ A\ B\ f\ (p\ 1)\ [\ \alpha \mapsto t\ 1\ ]}$$

duaF i A B f x := dua (i x) (A x) (B x) (f x)

dua is Covariant

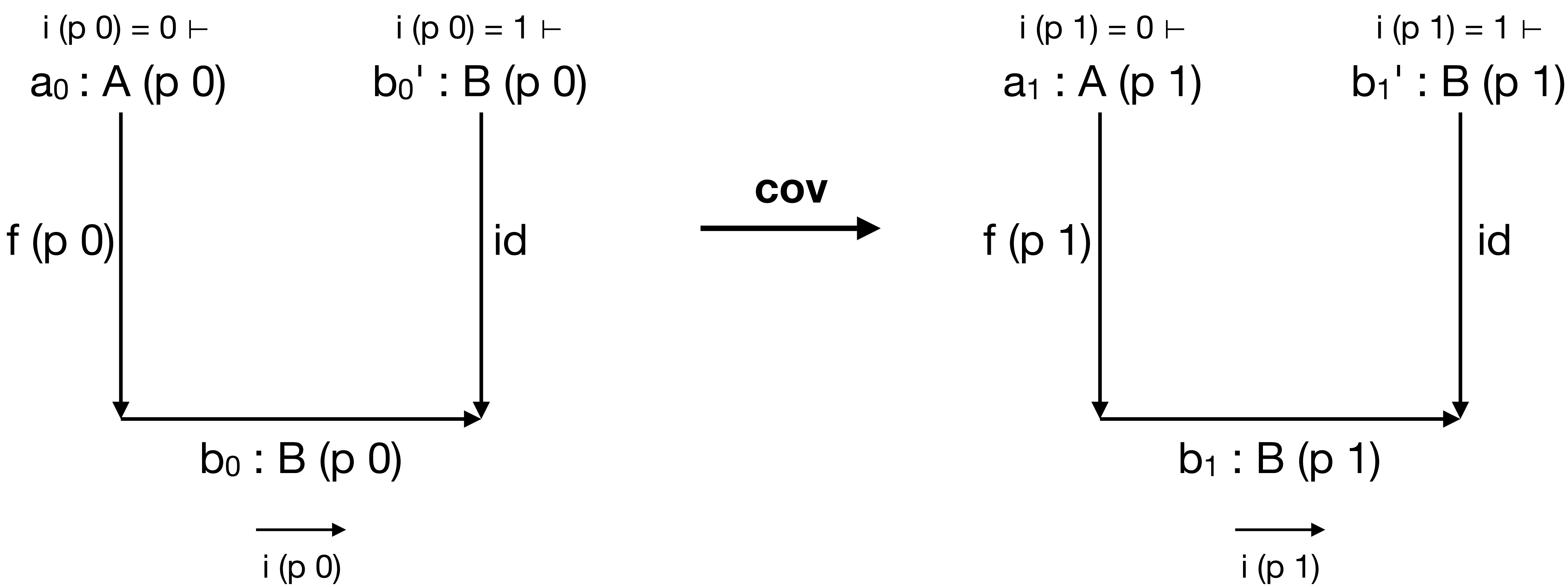
$$\frac{\Gamma : U \qquad i : \Gamma \to \mathbb{2} \qquad A : \Gamma \to \mathrm{UCov} \qquad B : \Gamma \to \mathrm{UCov} \\ f : (x : \Gamma) \to \mathrm{El}\ A\ x \to \mathrm{El}\ B\ x \qquad p : \mathbb{2} \to \Gamma \qquad b : \mathrm{duaF}\ i\ A\ B\ f\ (p\ 0)}{\mathrm{duaF}\ i\ A\ B\ f\ (p\ 1)}$$

(i.e. directed transport)

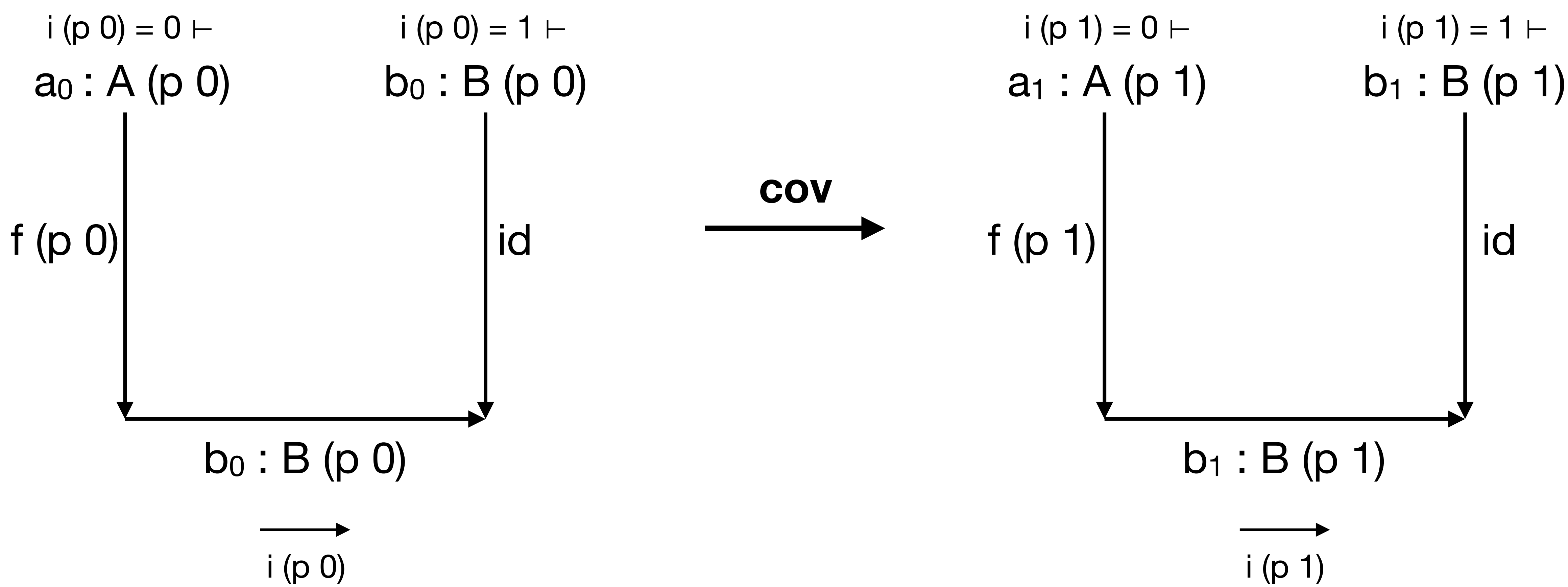
(generalized proof just carries along cofibration)

duaF i A B f x := dua (i x) (A x) (B x) (f x)

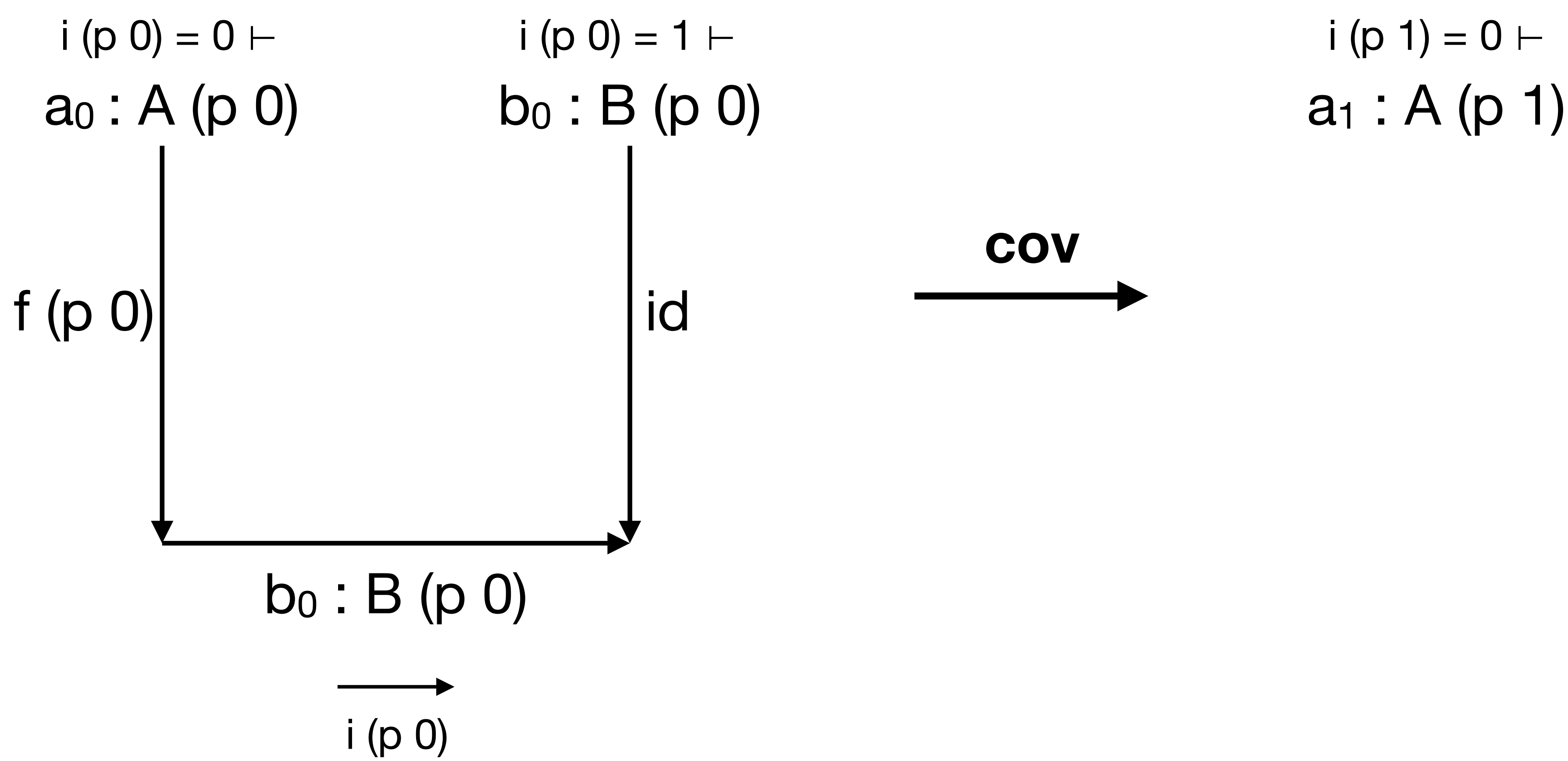
dua is Covariant



dua is Covariant



dua is Covariant



Monotonicity of $\mathbb{2} \to \mathbb{2}$ gives us that i (p 1) = 0 implies i (p j) = 0 for every j : $\mathbb{2}$

dua is Covariant

Given $\forall\, j : \mathbb{2}\,,\ i\,(p\ j) = 0$:

$$a_0 : A\,(p\ 0) \xrightarrow{\textbf{cov}} a_1 : A\,(p\ 1)$$

$$a_1 := \mathrm{cov}^{0 \mapsto 1}\ A\ p\ a_0 : A\,(p\ 1)$$

dua is Covariant

$$
\begin{tikzcd}
\substack{i\,(p\,0) = 0\,\vdash \\ a_0 : A\,(p\,0)} \arrow[d, "f\,(p\,0)"'] & \substack{i\,(p\,0) = 1\,\vdash \\ b_0 : B\,(p\,0)} \arrow[d, "id"] \\
{} \arrow[r, "b_0 : B\,(p\,0)"'] & {}
\end{tikzcd}
\quad \xrightarrow{\textbf{cov}} \quad
\begin{tikzcd}
\substack{i\,(p\,1) = 0\,\vdash \\ cov^{0\mapsto 1}\,A\;p\;a_0 : A\,(p\,1)} \arrow[d, "f\,(p\,1)"'] & \substack{i\,(p\,1) = 1\,\vdash \\ b_1 : B\,(p\,1)} \arrow[d, "id"] \\
{} \arrow[r, "b_1 : B\,(p1)"'] & {}
\end{tikzcd}
$$

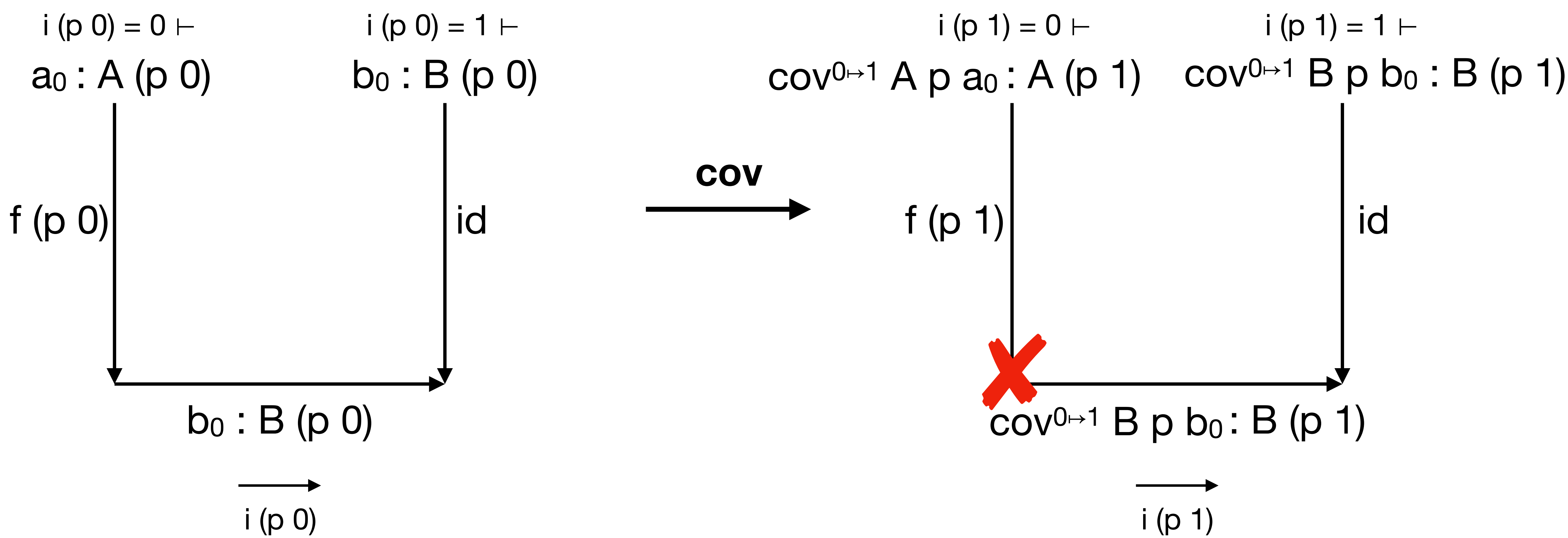
$$\xrightarrow[\;i\,(p\,0)\;]{} \qquad\qquad\qquad\qquad \xrightarrow[\;i\,(p\,1)\;]{}$$

dua is Covariant

$$b_0 : B\ (p\ 0) \xrightarrow{\textbf{cov}} b_1\ : B\ (p1)$$

$$b_1 := cov^{0 \mapsto 1}\ B\ p\ b_0 : B\ (p\ 1)$$

dua is Covariant



dua is Covariant

$$
\begin{tikzcd}
{\substack{i\,(p\,0) = 0 \vdash \\ a_0 : A\,(p\,0)}} \arrow[d, "f\,(p\,0)"'] & {\substack{i\,(p\,0) = 1 \vdash \\ b_0 : B\,(p\,0)}} \arrow[d, "id"] & & {\substack{i\,(p\,1) = 0 \vdash \\ \mathrm{cov}^{0\mapsto 1}\,A\,p\,a_0 : A\,(p\,1)}} \arrow[d, "f\,(p\,1)"'] & {\substack{i\,(p\,1) = 1 \vdash \\ b_1 : B\,(p\,1)}} \arrow[d, "id"] \\
{} \arrow[r, "b_0 : B\,(p\,0)"'] & {} \arrow[r, phantom, "\xrightarrow{\textbf{cov}}"] & {} & {} \arrow[r, "b_1 : B\,(p1)"'] & {}
\end{tikzcd}
$$

$\xrightarrow[i\,(p\,0)]{}$ $\xrightarrow[i\,(p\,1)]{}$

dua is Covariant

$$b_0 : B\,(p\;0)[\;i\,(p\;0) = 0 \mapsto f\,(p\;0)\;\;a_0] \xrightarrow{\textbf{cov}} b_1\; : B\,(p1)\;\;[\;i\,(p\;1) = 0 \mapsto f\,(p\;1)\;(cov^{0\mapsto 1}\;A\;p\;a_0)]$$

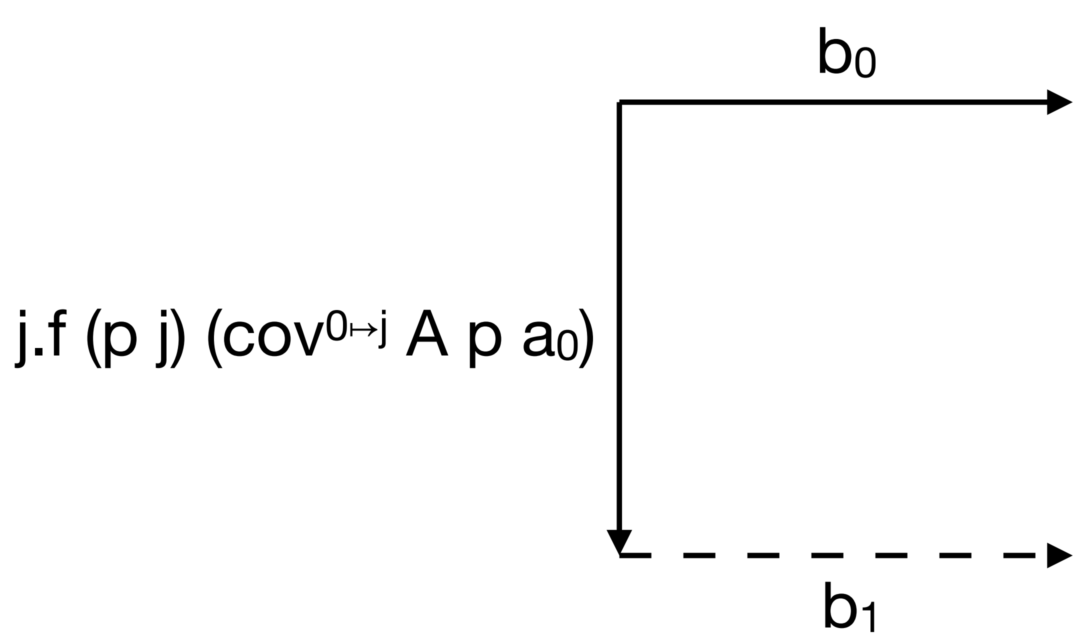
$$\begin{tikzcd}
f\,(p\;0)\;a_0 \arrow[r, "b_0"] & {} \\
f\,(p\;1)\;(cov^{0\mapsto 1}\;A\;p\;a_0) \arrow[r, dashed, "b_1"'] & {}
\end{tikzcd}$$

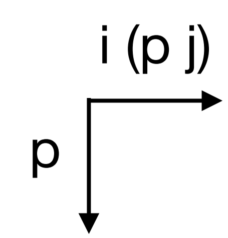
$$\begin{tikzcd}
\cdot \arrow[r, "i\,(p\;j)"] \arrow[d, "p"'] & {} \\
{} &
\end{tikzcd}$$

dua is Covariant

$$b_0 : B\ (p\ 0)[\ i\ (p\ 0) = 0 \mapsto f\ (p\ 0)\ \ a_0] \xrightarrow{\textbf{cov}} b_1\ : B\ (p1)\ \ [\ i\ (p\ 1) = 0 \mapsto f\ (p\ 1)\ (cov^{0\mapsto 1}\ A\ p\ a_0)]$$

$$b_1 := \ cov^{0\mapsto 1}\ B\ [i\ (p\ 1) = 0 \mapsto j.f\ (p\ j)\ (cov^{0\mapsto j}\ A\ p\ a_0)]\ p\ b_0$$





Again, monotonicity allows us to satisfy the boundary constraint

dua is Covariant

$$\begin{tikzcd}[column sep=large, row sep=huge]
\substack{i\,(p\,0) = 0 \,\vdash \\ a_0 : A\,(p\,0)} \arrow[d, "f\,(p\,0)"'] & \substack{i\,(p\,0) = 1 \,\vdash \\ b_0 : B\,(p\,0)} \arrow[d, "\mathrm{id}"] & & \substack{i\,(p\,1) = 0 \,\vdash \\ \mathrm{cov}^{0 \mapsto 1}\,A\,p\,a_0 : A\,(p\,1)} \arrow[d, "f\,(p\,1)"'] & \substack{i\,(p\,1) = 1 \,\vdash \\ \mathrm{cov}^{0 \mapsto 1} \ldots : B\,(p\,1)} \arrow[d, "\mathrm{id}"] \\
{} \arrow[r, "b_0 : B\,(p\,0)"'] & {} \arrow[r, phantom, "\xrightarrow{\textbf{cov}}"] & {} & {} \arrow[r, "\mathrm{cov}^{0 \mapsto 1}\,B\,[i\,(p\,1) = 0 \,\mapsto\, j.f\,(p\,j)\,(\mathrm{cov}^{0 \mapsto j}\,A\,p\,a_0)]\,p\,b_0 \,:\, B\,(p\,1)"'] & {}
\end{tikzcd}$$

$\xrightarrow{}$ i (p 0) $\qquad\qquad$ $\xrightarrow{}$ i (p 1)

Directed Univalence for UCov

- We have proven 3/4 of directed univalence for UCov
  - Functions between $\mathrm{Hom}_{\mathrm{UCov}}$ A B and El A → El B
  - El A → El B is a retract of $\mathrm{Hom}_{\mathrm{UCov}}$ A B
- Remaining inverse seems difficult
  - Unclear if Riehl-Sattler proof for this is constructive
- Assuming full dua, we can show UCov is Segal

Using Directed Univalence

- Application: formal (meta)theory of programming languages/computational structures
- Goal: Represent syntax so is "invariant" under weakening/renaming/substitution/transition/evaluation/etc...
- Example:
  - Contexts are Type/Category where morphisms are structural rules/substitutions
  - Term : Context → Set is functorial

Using Directed Univalence

$$
\begin{aligned}
&\textbf{data}\ \mathsf{Ctx} : \mathsf{USegal}\ \textbf{where} \\
&\quad \bullet : \mathsf{Ctx} \\
&\quad \mathsf{ext} : \mathsf{Ctx} \to \mathsf{Ctx} \\
&\quad \mathsf{wk} : (\Gamma : \mathsf{Ctx}) \to \mathsf{Hom}\ \Gamma\ (\mathsf{ext}\ \Gamma)
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{Ctx\text{-}rec} : \\
&\quad \{\mathsf{A} : \mathsf{USegal}\} \\
&\quad (\mathsf{c}_0 : \mathsf{A}) \\
&\quad (\mathsf{c}_1 : \mathsf{A} \to \mathsf{A}) \\
&\quad (\mathsf{c}_2 : (\mathsf{a} : \mathsf{A}) \to \mathsf{Hom}\ \mathsf{a}\ (\mathsf{c}_1\ \mathsf{a})) \\
&\quad \to \mathsf{Ctx} \to \mathsf{A}
\end{aligned}
$$

Important A is Segal to incorporate freely added composites

Using Directed Univalence

$$\mathsf{Var} \;:\; \mathsf{Ctx} \to \mathsf{UCov}$$
$$\mathsf{Var} \;=\; \mathsf{Ctx\text{-}rec}\ \bot\ (\lambda\, \mathsf{X} \to \top + \mathsf{X})\ (\lambda\, \mathsf{X} \to \mathsf{dua}\ \{\mathsf{X}\}\ \{\top + \mathsf{X}\}\ \mathsf{inr})$$

$$\begin{aligned}
&\mathsf{Ctx\text{-}rec} \;: \\
&\quad \{\mathsf{A} \;:\; \mathsf{USegal}\} \\
&\quad (\mathsf{c}_0 \;:\; \mathsf{A}) \\
&\quad (\mathsf{c}_1 \;:\; \mathsf{A} \to \mathsf{A}) \\
&\quad (\mathsf{c}_2 \;:\; (\mathsf{a} \;:\; \mathsf{A}) \to \mathsf{Hom}\ \mathsf{a}\ (\mathsf{c}_1\ \mathsf{a})) \\
&\quad \to \mathsf{Ctx} \to \mathsf{A}
\end{aligned}$$

Using Directed Univalence

Tm t := var x
| λ t
| app t t'

$$\mathsf{TmF} : (\mathsf{Ctx} \to \mathsf{UCov}) \to (\mathsf{Ctx} \to \mathsf{UCov})$$
$$\mathsf{TmF}\ \mathsf{T}\ \Gamma = \mathsf{Var}\ \Gamma + \mathsf{T}\ (\mathsf{ext}\ \Gamma) + \mathsf{T}\ \Gamma \times \mathsf{T}\ \Gamma$$

$$\mathsf{Tm} : \mathsf{Ctx} \to \mathsf{UCov}$$
$$\mathsf{Tm} = \mu\ \mathsf{TmF}$$

1. UCov is closed under +
2. UCov is closed under ×
3. UCov is closed under fixpoints of of polynomial functors?

Using Directed Univalence

Consider the term:

λ (λ app (var 1) (var 0))

Using this infrastructure, we automatically get:

dcoe Tm (wk ·) (λ (λ app (var 1) (var 0)))
= λ (λ app (var 2) (var 1))