FINITE INVERSE CATEGORIES AS SIGNATURES

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Abstract. We define a simple type theory and prove that its well-formed contexts correspond exactly to finite inverse categories.

1. Introduction

A finite inverse category (*fic*) is a finite, skeletal category with no non-identity endomorphisms. Fics are interesting from the point of view of dependent type theory because they correspond to contexts of dependently-typed data, and especially the contexts that are useful for formalizing mathematics in the style of HoTT/UF. The basic idea is well-illustrated by the following simple example.

Example 1.1. Consider the following fic

$$\mathcal{L}_{\mathrm{rg}} =_{\mathrm{df}} \qquad \begin{array}{c} I \\ {\scriptstyle i}\downarrow \\ A \\ {\scriptstyle c}\downarrow\;\downarrow{\scriptstyle d} \\ O \end{array}$$

subject to the relation $di = ci$. The data in $\mathcal{L}_{\mathrm{rg}}$ corresponds to the following collection of data in dependent type theory, writing $\mathcal{U}$ for a universe of types:

$$\begin{aligned} &O\colon \mathcal{U} \\ &A\colon O \times O \to \mathcal{U} \\ &I\colon (\Sigma\,(x\colon O)\; A(x,x)) \to \mathcal{U} \end{aligned}$$

In this paper, we define a certain "mini-type theory" whose well-formed contexts correspond precisely to fics. Another way to look at this "mini-type theory" is that it encodes a quotient-inductive-inductive type (QIIT) of all fics, similar to what is done in [AK16].

Both perspectives are important: the former allows us to work "externally" in order to define interpretations of fics in dependent type theory (as e.g. described in [Tse16]), whereas the latter allows us to implement such an interpretation inside type theory, as we intend to do in [WT].

However, this paper is independent of either of these perspectives. Its aim is to define a certain simplified type theory together with its categorical semantics and

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prove ("externally") that the well-formed contexts of this type theory correspond precisely to finite inverse categories as usually defined. This is done in Theorem 4.7, the main result of this paper.

Related Work. The observation that finite inverse categories can play the role of a dependently-typed syntax was perhaps first made by Makkai in his work on First Order Logic With Dependent Sorts (FOLDS) [Mak95], with Cartmell's work [Car86] as an important precursor. The connection between FOLDS and MLTT/HoTT has been pursued in [Tse16, ANS14]. Also related is Palmgren's work in [Pal16] where he develops a general notion of dependently-typed first-order logic, independent from our own, connecting some special cases to FOLDS, and thereby to fics. Finite inverse categories can also be understood as special cases of Reedy categories, and so in that form they have been related to homotopy (type) theory, and studied extensively in [Shu15a, Shu15b].

Outline. In Section 2, we define the syntax and rules of the type theory $\mathbb{T}_{\text{fic}}$. In Section 3, we define a categorical semantics for $\mathbb{T}_{\text{fic}}$ and prove soundness. Finally, in Section 4 we prove that the well-formed contexts (or "signatures") of $\mathbb{T}_{\text{fic}}$ correspond exactly to finite inverse categories (Theorem 4.7).

2. The Theory $\mathbb{T}_{\text{FIC}}$

Definition 2.1 (Syntax of $\mathbb{T}_{\text{fic}}$)**.** We assume fixed countably infinite disjoint sets of *variables* $x, y, ... \in Var$ and *names* $A, K, ... \in SN$. The syntax of $\mathbb{T}_{\text{fic}}$ consists of *signatures*, *contexts*, and *substitutions* defined as follows:

Ψ, Φ	$::=$		Signatures
	$\mid$	$\bullet$	empty signature
	$\mid$	$\Psi, A : \Gamma$	signature extension, $A \in SN$
$\Gamma,\ \Delta$	$::=$		Contexts
	$\mid$	$\bullet$	empty context
	$\mid$	$\Gamma, x : A\, \sigma$	context extension, $x \in Var$
σ, δ	$::=$		Substitutions
	$\mid$	ϵ_Γ	empty substitution
	$\mid$	id_Γ	identity substitution
	$\mid$	wk_σ	weakening
	$\mid$	contr_σ	contraction
	$\mid$	$\mathsf{sw}_{\sigma,\delta}$	swapping
	$\mid$	$\sigma \cdot \delta$	composition
	$\mid$	σ, δ	substitution extension

As usual, we consider these expressions up to α-equivalence, i.e. up to consistent renaming of variables and names.

Definition 2.2 (Judgments of $\mathbb{T}_{\text{fic}}$)**.** We have the following judgments built out of our syntax, displayed together with their intended categorical interpretation, for ease of understanding:

$\Psi \vdash$	"Ψ is a finite inverse category"
$\Psi \vdash \Gamma$	"Γ is a finite functor $\Psi \to \mathbf{Set}$"
$\Psi \vdash \sigma \colon \Gamma \Rightarrow \Delta$	"σ is a natural transformation $\Delta \to \Gamma$"
$\Psi \vdash \sigma \equiv \delta : \Gamma \Rightarrow \Delta$	"$\sigma = \delta$ as natural transformations"

Definition 2.3 (Rules of $\mathbb{T}_{\text{fic}}$)**.** The rules of $\mathbb{T}_{\text{fic}}$ are the following, grouped according the form of the judgment in their conclusion:

$\boxed{\Psi \vdash}$ Well-Formed Signatures

$$\frac{}{\bullet \vdash}\ \textsc{Sig-}\bullet$$

$$\frac{\Psi \vdash \quad \Psi \vdash \Gamma}{\Psi, A : \Gamma \vdash}\ \textsc{Sig-ext}$$

$\boxed{\Psi \vdash \Gamma}$ Well-Formed Contexts

$$\frac{\Psi \vdash}{\Psi \vdash \bullet}\ \textsc{Con-}\bullet$$

$$\frac{\Psi, A : \Delta, \Psi' \vdash \Gamma \quad \Psi, A : \Delta, \Psi' \vdash \sigma : \Gamma \Rightarrow \Delta}{\Psi, A : \Delta, \Psi' \vdash \Gamma,\, x : A\, \sigma}\ \textsc{Con-ext}$$

$\boxed{\Psi \vdash \sigma : \Gamma \Rightarrow \Delta}$ Substitutions

$$\frac{}{\Psi \vdash \epsilon_\Gamma : \Gamma \Rightarrow \bullet}\ \textsc{Sub-}\epsilon$$

$$\frac{\Psi \vdash\ \Gamma}{\Psi \vdash \mathsf{id}_\Gamma : \Gamma \Rightarrow \Gamma}\ \textsc{Sub-}\mathsf{id}$$

$$\frac{\Psi \vdash \Gamma, x : A\, \sigma}{\Psi \vdash \mathsf{wk}_\sigma : \big(\Gamma,\, x : A\, \sigma\big) \Rightarrow \Gamma}\ \textsc{Sub-}\mathsf{wk}$$

$$\frac{\Psi \vdash \Gamma, x : A\sigma}{\Psi \vdash \mathsf{contr}_\sigma : \big(\Gamma,\, x : A\, \sigma\big) \Rightarrow \big(\Gamma,\, x : A\, \sigma,\, y : A'\, (\mathsf{wk}_\sigma \cdot \sigma)\big)}\ \textsc{Sub-}\mathsf{contr}$$

$$\frac{\Psi \vdash \sigma : \Gamma \Rightarrow \Delta \quad \Psi \vdash \delta : \Gamma \Rightarrow \Theta}{\Psi \vdash \mathsf{sw}_{\sigma,\delta} : \big(\Gamma,\, x : A\, \sigma,\, y : K\, (\mathsf{wk}_\sigma \cdot \delta)\big) \Rightarrow \big(\Gamma,\, y : K\, \delta,\, x : A\, (\mathsf{wk}_\delta \cdot \sigma)\big)}\ \textsc{Sub-}\mathsf{sw}$$

$$\frac{\Psi \vdash \sigma : \Gamma \Rightarrow \Delta \quad \Psi \vdash \delta : \Delta \Rightarrow \Theta}{\Psi \vdash \sigma \cdot \delta : \Gamma \Rightarrow \Theta} \quad \textsc{Sub-}\cdot$$

$$\frac{\Psi \vdash \sigma : \Gamma \Rightarrow \Delta \quad \Psi \vdash \ \Gamma, x : A\,(\sigma \cdot \delta)}{\Psi \vdash \sigma, \delta : \big(\Gamma, x : A\,(\sigma \cdot \delta)\big) \Rightarrow \Delta, y : A\,\delta} \quad \textsc{Sub-ext}$$

As usual, we always require that a newly-introduced variable or name is fresh for (i.e. does not appear in) the context or signature expression it is attached to.

The congruences of $\mathbb{T}_{\text{fic}}$ are the following, meant to hold whenever their constituents have been derived and it makes sense to compare them:

(ϵid)	$\epsilon_\bullet \equiv \mathsf{id}_\bullet$
($\epsilon\epsilon$)	$\sigma \cdot \epsilon_\Delta \equiv \epsilon_\Gamma$
(Lid)	$\mathsf{id}_\Gamma \cdot \sigma \equiv \sigma$
(Rid)	$\sigma \cdot \mathsf{id}_\Gamma \equiv \sigma$
(idext)	$\mathsf{id}_\Gamma, \sigma \equiv \mathsf{id}_{\Gamma,\sigma}$
(assoc)	$\delta \cdot (\sigma \cdot \tau) \equiv (\delta \cdot \sigma) \cdot \tau$
(contrwk)	$\mathsf{contr}_\sigma \cdot \mathsf{wk}_{\mathsf{wk}_\sigma \cdot \sigma} \equiv \mathsf{id}_{\Gamma,\sigma}$
(extcomp)	$(\sigma \cdot \tau), \delta \equiv (\sigma, \tau \cdot \delta) \cdot (\tau, \delta)$
(extcompwk)	$\sigma, \delta \cdot \mathsf{wk}_\delta \equiv \mathsf{wk}_{\sigma\cdot\delta} \cdot \sigma$
(swsw)	$\mathsf{sw}_{\sigma,\delta} \cdot \mathsf{sw}_{\delta,\sigma} \equiv \mathsf{id}_{\Gamma,\sigma,\mathsf{wk}_\sigma\cdot\delta}$

Finally, we have the usual structural rules that assert that derivably congruent expressions can be substituted for each other in any other expression.

3. Categorical Semantics of $\mathbb{T}_{\text{fic}}$

Terminology. A category is *finite* if its set of morphisms (and hence also of objects) is finite. A category is *skeletal* if its only isomorphisms are the identity arrows. We say that a functor $F : \mathcal{C} \to \mathbf{Set}$ is *finite* if $F(a)$ is a finite set for all objects a in $\mathcal{C}$.

Definition 3.1. A **finite inverse category** (*fic*) is a finite skeletal category with no non-identity endomorphisms.

Definition 3.2. Let $\mathcal{L}$ be a fic. An $\mathcal{L}$**-context** is a finite functor $\mathcal{L} \to \mathbf{Set}$. We write $\mathbf{Con}(\mathcal{L})$ for the full subcategory of $\mathbf{Set}^{\mathcal{L}}$ consisting of the finite functors.

Notation. For a fic $\mathcal{L}$ and $K \in \mathrm{ob}\,\mathcal{L}$ and $f\colon K \to K'$ we write K_f for K'. We write $K \downarrow \mathcal{L}$ for the proper cosieve on K in $\mathcal{L}$, i.e. the set of all non-identity maps with K as their domain. We write ∂K for the reduced Yoneda functor at K, i.e. $\partial K(A) = \mathbf{y}K(A)$ for all $A \neq K$ but $\partial A(A) = \varnothing$. Both ∂K and $\mathbf{y}K$ are obviously $\mathcal{L}$-contexts. Similarly for any $\mathcal{L}$-context G and $x \in G(K)$ we write ∂x for the "boundary" natural transformation $\partial K \to G$ which maps $y \mapsto g(x)$ for every $g : K \to K'$ and $y \in G(K')$. We write 1 for a generic singleton set and for convenience we will also often write 1 for its unique element. Also for convenience we will assume that any fic $\mathcal{L}$ we consider is such that $\mathrm{ob}\,\mathcal{L} \subset SN$ and the non-identity morphisms in $\mathcal{L}$ are a subset of Var, which allows us to use the same notation for variables in $\mathbb{T}_{\text{fic}}$ and for arrows of fics.

We will now give a semantics to $\mathbb{T}_{\text{fic}}$ where signatures are interpreted as finite inverse categories $\mathcal{L}$, contexts as $\mathcal{L}$-contexts and substitutions as natural transformations between $\mathcal{L}$-contexts (i.e. morphisms in $\mathbf{Con}(\mathcal{L})$). To do so we will first define certain operations on fics and their contexts.

Definition 3.3 (Fic extension)**.** Let $\mathcal{L}$ be a fic. For any $\mathcal{L}$-context $G : \mathcal{L} \to \mathbf{Set}$ the **fic extension** of $\mathcal{L}$ by G is a fic $\mathcal{L} * G$ defined as follows:

- $\operatorname{ob}(\mathcal{L} * G) = (\operatorname{ob}\mathcal{L}) \amalg 1$
- $$\mathcal{L} * G(X, Y) = \begin{cases} \mathcal{L}(X, Y) & \text{if } X, Y \neq 1 \\ 1_1 & \text{if } X = Y = 1 \\ \varnothing & \text{if } Y = 1, X \neq 1 \\ G(Y) & \text{if } X = 1, Y \neq 1 \end{cases}$$
- Let $f \in \mathcal{L} * G(X, Y)$ and $g \in \mathcal{L} * G(Y, Z)$. We define the composition $g \circ f \in \mathcal{L} * G(X, Z)$ as follows:
$$g \circ f = \begin{cases} g \circ_{\mathcal{L}} f & \text{if } X, Y, Z \neq 1 \\ g & \text{if } f = 1_1 \\ G(g)(f) & \text{if } X = 1, Y \neq 1, Z \neq 1 \end{cases}$$

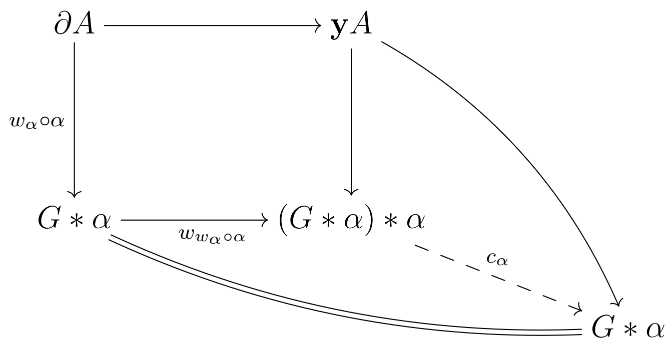
We write $\mathcal{L} *_A G$ when we want to give a specific name A to the new object introduced in $\mathcal{L} * G$ and we always assume that $A \notin \operatorname{ob}\mathcal{L}$.

Remark 3.4. What we call a "fic extension" is a special case of what has also been called a *collage* or a *cograph*.

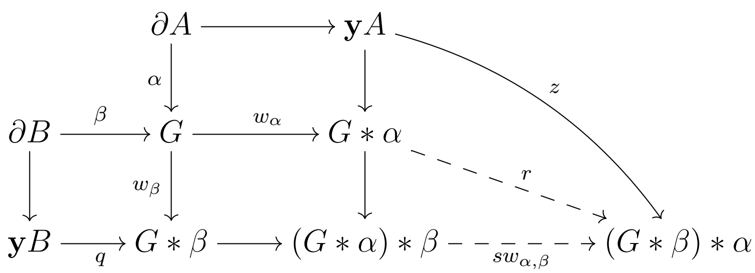
Definition 3.5 (Context Extension, Weakening, Contraction, Swap)**.** Let $\mathcal{L}$ be a fic and $G : \mathcal{L} \to \mathbf{Set}$ an $\mathcal{L}$-context. For any object A of $\mathcal{L}$ and any natural transformation $\alpha : \partial A \to G$ in $\mathbf{Con}(\mathcal{L})$ the **context extension** of G by α is the $\mathcal{L}$-context $G * \alpha$ given as the following pushout

$$\begin{array}{ccc} \partial A & \longrightarrow & \mathbf{y}A \\ {\scriptstyle\alpha}\downarrow & & \downarrow \\ G & \xrightarrow[w_\alpha]{} & G * \alpha \end{array}$$

We write $G *_x \alpha$ when we want to give a specific name x to the map $\mathbf{y}A \to G * \alpha$. We call w_α the **weakening** of α. The **contraction** c_α of α is defined as the following unique map induced by the universal property of the pushout



Finally, assume we are also given $\beta : \partial B \to G$ where B is an object of $\mathcal{L}$ such that there is no non-identity map between B and A in $\mathcal{L}$. Then we denote by $sw_{\alpha,\beta}$ the following unique map given by the universal property of the relevant pushout square:

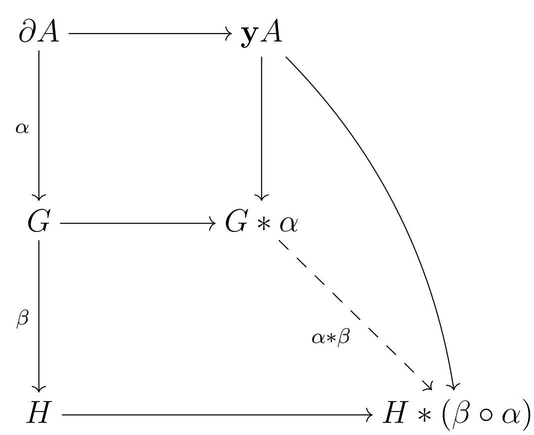


In more detail, we have the unique map $r = [w_{w_\beta \circ \alpha} \circ w_\beta, z]$ and we then define

$$sw_{\alpha,\beta} =_{\text{df}} [w_{w_\beta \circ \alpha} \circ q, r]$$

where z and q are the coprojections of the respective pushouts. We call $sw_{\alpha,\beta}$ the **swap** of α and β.

Definition 3.6 (Substitution Extension)**.** Let $\mathcal{L}$ be a fic, $G, H : \mathcal{L} \to \mathbf{Set}$ be $\mathcal{L}$-contexts and A an object of $\mathcal{L}$. For any natural transformations $\alpha : \partial A \to G$ and $\beta : G \to H$ in $\mathbf{Con}(\mathcal{L})$ the **substitution extension** of α by β is the natural transformation $\alpha * \beta$ given by the following diagram



Definition 3.7 (Interpretation)**.** We define a partial interpretation function $[\![-]\!]$ by induction on the complexity of the raw syntax, as follows:

- $[\![\bullet]\!] = \emptyset$, the empty fic
- $[\![\Psi, A : \Gamma]\!] = [\![\Psi]\!] *_A [\![\Gamma]\!]$, if $[\![\Gamma]\!]$ is a $[\![\Psi]\!]$-context and $A \notin \mathrm{ob}\, [\![\Psi]\!]$
- $[\![\bullet]\!] = \emptyset$, the empty $\mathcal{L}$-context for some fic $\mathcal{L}$
- $[\![\Gamma, x : A\,\sigma]\!] = [\![\Gamma]\!] *_x [\![\sigma]\!]$, if $[\![\Gamma]\!]$ is an $\mathcal{L}$-context for some fic $\mathcal{L}$, $[\![\sigma]\!] : [\![\Delta]\!] \to [\![\Gamma]\!]$ is a natural transformation for some $\mathcal{L}$-context $[\![\Psi]\!]$ such that $[\![\Delta]\!] = \partial A$ and $[\![\sigma]\!] = \partial x$
- $[\![\epsilon_\Gamma]\!] = \emptyset \xrightarrow{!_{[\![\Gamma]\!]}} [\![\Gamma]\!]$
- $[\![\mathsf{id}_\Gamma]\!] = [\![\Gamma]\!] \xrightarrow{1_{[\![\Gamma]\!]}} [\![\Gamma]\!]$
- $[\![\mathsf{wk}_\sigma]\!] = w_{[\![\sigma]\!]}$
- $[\![\mathsf{contr}_\sigma]\!] = c_{[\![\sigma]\!]}$
- $[\![\mathsf{sw}_{\sigma,\delta}]\!] = sw_{[\![\sigma]\!],[\![\delta]\!]}$
- $[\![\sigma \cdot \delta]\!] = [\![\sigma]\!] \circ [\![\delta]\!]$
- $[\![\sigma, \delta]\!] = [\![\sigma]\!] * [\![\delta]\!]$

Given the interpretation of the raw syntax we define the interpretation of the judgments as follows:

- $[\![\Psi \vdash]\!]$ if $[\![\Psi]\!]$ is a fic.
- $[\![\Psi \vdash \Gamma]\!]$ if $[\![\Psi]\!]$ is a fic and $[\![\Gamma]\!]$ is a $[\![\Psi]\!]$-context.
- $[\![\Psi \vdash \sigma : \Gamma \Rightarrow \Delta]\!]$ if $[\![\Psi]\!]$ is a fic, $[\![\Gamma]\!]$, $[\![\Delta]\!]$ are $[\![\Psi]\!]$-contexts, and $[\![\sigma]\!]$ is a natural transformation $[\![\Delta]\!] \to [\![\Gamma]\!]$.
- $[\![\Psi \vdash \sigma \equiv \tau : \Gamma \Rightarrow \Delta]\!]$ if $[\![\Psi]\!]$ is a fic, $[\![\Gamma]\!]$, $[\![\Delta]\!]$ are $[\![\Psi]\!]$-contexts, $[\![\sigma]\!], [\![\tau]\!]$ are natural transformations $[\![\Delta]\!] \to [\![\Gamma]\!]$, and $[\![\sigma]\!] = [\![\tau]\!]$ as natural transformations.

We say that a judgment $\mathcal{J}$ is *derivable* if there is a $\mathbb{T}_{\text{fic}}$-derivation concluding with $\mathcal{J}$. We now have the following soundness theorem, given in the style of Streicher [Str12], stating that the above-defined partial interpretation function is total on the derivable judgments of $\mathbb{T}_{\text{fic}}$.

Theorem 3.8 (Soundness)**.** *If $\mathcal{J}$ is derivable then $[\![\mathcal{J}]\!]$ is defined.*

Proof. We proceed by induction on the complexity of derivations, taking the rules and congruences of $\mathbb{T}_{\text{fic}}$ in turn.

(Sig-$\bullet$): Since $[\![\bullet]\!]$ is the empty fic, this is immediate since the empty fic is in particular a fic.

(Sig-ext): Assume that $[\![\Psi]\!]$ is a fic and that $[\![\Gamma]\!]$ is a $[\![\Psi]\!]$-context and $A \notin \mathrm{ob}\, [\![\Psi]\!]$. Then $[\![\Psi]\!] *_A [\![\Gamma]\!] = [\![\Psi, A : \Gamma]\!]$ is a fic, as required.

(Con-$\bullet$): Since $[\![\bullet]\!]$ is the empty context, this is immediate since the empty context is in particular a context for any fic $[\![\Psi]\!]$.

(Con-ext): Assume that $[\![\Psi, A : \Delta, \Psi']\!]$ is a fic, $[\![\Gamma]\!]$ a $[\![\Psi, A : \Delta, \Psi']\!]$-context and that $[\![\sigma]\!] : [\![\Delta]\!] \to [\![\Gamma]\!]$ is a natural transformation. These assumptions in particular imply that $[\![\Delta]\!] \cong \partial A$ for A the new object in $[\![\Psi, A : \Delta]\!] = [\![\Psi]\!] *_A [\![\Delta]\!]$. Therefore, the operation $[\![\Gamma]\!] *_x [\![\sigma]\!]$ is defined for any $x \notin [\![\Gamma]\!](A)$ which means that $[\![\Gamma, x : A\sigma]\!]$ is defined, as required.

(Sub-ϵ): If $[\![\Gamma]\!]$ is an $[\![\Psi]\!]$-context then there is a unique natural transformation $! = \epsilon_{[\![\Gamma]\!]} : \varnothing \to [\![\Gamma]\!]$, as required.

($\mathsf{Sub\text{-}id}$): Analogous, and omitted.

($\mathsf{Sub\text{-}wk}$): Since we assume $[\![\Psi \vdash \Gamma,\ x : A\sigma]\!]$ is defined this means that we assume in particular that $[\![\Gamma]\!]$ is a $[\![\Psi]\!]$-context and that $[\![\Gamma,\ x : A\sigma]\!]$ is defined and therefore that $[\![\sigma]\!]$ is defined. But $w_{[\![\sigma]\!]} = [\![\mathsf{wk}_\sigma]\!]$ which means exactly that $[\![\Psi \vdash \mathsf{wk}_\sigma : \Gamma,\ x : A\sigma \Rightarrow \Gamma]\!]$ is defined, as required.

($\mathsf{Sub\text{-}contr}$): Analogous, and omitted.

($\mathsf{Sub\text{-}sw}$): Analogous, and omitted.

(Sub-$\cdot$): Immediate.

($\mathsf{Sub\text{-}ext}$): From the assumptions we know that we have a fic $[\![\Psi]\!]$, natural transformations $[\![\sigma]\!] : [\![\Delta]\!] \to [\![\Gamma]\!]$ and $[\![\Delta]\!] : \partial A \to [\![\Delta]\!]$ for some $A \in [\![\Psi]\!]$. Therefore the operation $[\![\sigma]\!] * [\![\delta]\!]$ is defined which means exactly that $[\![\sigma, \delta]\!]$ is defined, as required.

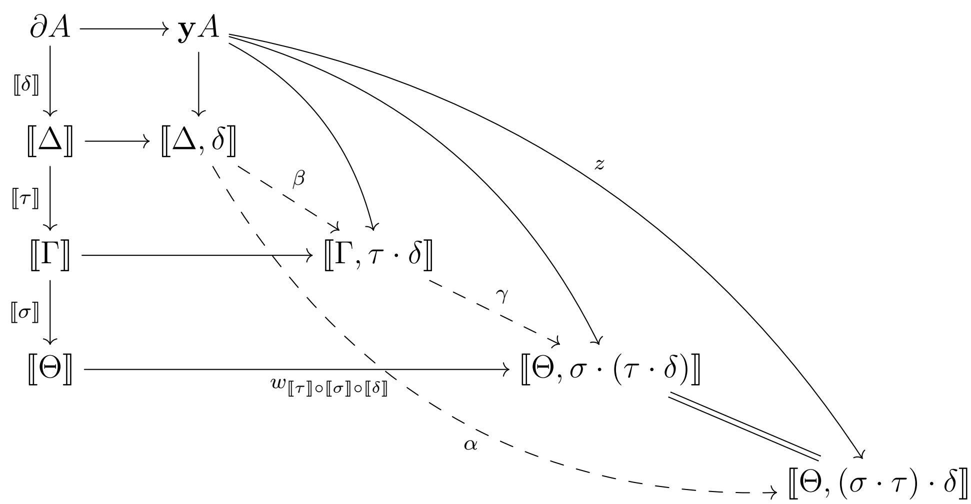
It remains to show that the congruences are true. ($\epsilon\mathsf{id}$), ($L\mathsf{id}$), ($R\mathsf{id}$) and (assoc) are immediate and ($\mathsf{contrwk}$), ($\mathsf{extcompwk}$) are straightforward from the definitions (they simply assert the commutativity of certain triangles induced by canonical maps from a pushout). For ($\mathsf{extcomp}$), assuming that everything is defined, we have

$$\begin{aligned} [\![(\sigma \cdot \tau), \delta]\!] &= [\![\sigma \cdot \tau]\!] * [\![\delta]\!] \\ &= ([\![\sigma]\!] \circ [\![\tau]\!]) * [\![\delta]\!] \\ &=_{\mathrm{df}} \alpha \end{aligned}$$

and

$$\begin{aligned} [\![(\sigma, (\tau \cdot \delta)) \cdot (\tau, \delta)]\!] &= ([\![\sigma]\!] * ([\![\tau]\!] \circ [\![\delta]\!])) \circ ([\![\tau]\!] * [\![\delta]\!]) \\ &=_{\mathrm{df}} \gamma \circ \beta \end{aligned}$$

where $\beta =_{\mathrm{df}} [\![\tau]\!] * [\![\delta]\!]$ and $\gamma =_{\mathrm{df}} [\![\sigma]\!] * ([\![\tau]\!] \circ [\![\delta]\!])$. It suffices to show that $\alpha = \gamma \circ \beta$. To that end, consider the following diagram



It is easily seen that the diagram commutes which implies exactly that $\alpha = \gamma \circ \beta$ since both α and $\gamma \circ \beta$ satisfy the same properties as the unique map $[[\![\tau]\!] \circ [\![\sigma]\!] \circ w_{[\![\tau]\!] \circ [\![\sigma]\!] \circ [\![\delta]\!]}, z]$ induced by the pushout square for $[\![\Delta, \delta]\!]$. The remaining congruences (idext) and (swsw) follow by similar arguments, and are left to the reader. □

4. All fics are $\mathbb{T}_{\text{FIC}}$-signatures

We will now prove that any finite inverse category is isomorphic to the interpretation of a derivable $\mathbb{T}_{\text{fic}}$-signature. We require some preliminary definitions and lemmas.

Definition 4.1 (Length of a fic)**.** The **length** $l(\mathcal{L})$ of a fic $\mathcal{L}$ is the cardinality of $\mathrm{ob}\,\mathcal{L}$.

Definition 4.2 (Height of a sort)**.** The **height** $h(K)$ of $K \in \mathrm{ob}\mathcal{L}$ for a fic $\mathcal{L}$ is defined inductively as follows

$$h(K) = \begin{cases} 1 & \text{if } K \text{ is the domain only of } 1_K \\ \sup\limits_{f \in K \downarrow \mathcal{L}} h(K_f) + 1 & \text{otherwise} \end{cases}$$

Definition 4.3 (Height of an $\mathcal{L}$-context)**.** Let $\mathcal{L}$ be a fic. The **height** $h(G)$ of an $\mathcal{L}$-context G is given by

$$h(G) = \begin{cases} 0 & \text{if } G = \emptyset \\ \sup\limits_{\substack{A \in \mathrm{ob}\,\mathcal{L} \\ G(A) \neq \varnothing}} h(A) & \text{otherwise} \end{cases}$$

In other words, $h(G)$ is the maximum height among the sorts A in $\mathcal{L}$ whose image under G is non-empty.

We now prove that certain canonical isomorphisms are derivable in $\mathbb{T}_{\text{fic}}$. In order to state the lemma, we use the following useful abbreviation for an (a priori not necessarily derivable) substitution expression

$$\begin{aligned}\mathsf{sw}_{\sigma,\delta}(\epsilon_1,\ldots,\epsilon_n) =_{\text{df}}\ &\mathsf{sw}_{\sigma,\delta}\cdot\epsilon_1, (\mathsf{sw}_{\sigma,\delta},\epsilon_1)\cdot\epsilon_2, ((\mathsf{sw}_{\sigma,\delta},\epsilon_1),\epsilon_2)\cdot\epsilon_3,\ldots\\ &\ldots,((\mathsf{sw}_{\sigma,\delta},\epsilon_1),\ldots,\epsilon_{n-1}))\cdot\epsilon_n\end{aligned}$$

If $\Delta = x_1 : A_1\ \epsilon_1,\ldots,x_n : A_n\ \epsilon_n$ is a context expression then we will write simply $\mathsf{sw}_{\sigma,\delta}\Delta$ for the above expression.

Lemma 4.4. *Assume that*

$$\Gamma, x : A\ \sigma,\ y : B\ (\mathsf{wk}_\sigma\cdot\delta), \Delta$$

and

$$\Gamma, y : B\ \delta, x : A\ (\mathsf{wk}_\delta\cdot\sigma), \mathsf{sw}_{\sigma,\delta}\Delta$$

are derivable contexts, where $\Delta = x_1 : A_1\ \epsilon_1,\ldots,x_n : A_n\ \epsilon_n$. *Then*

$$\begin{aligned}\mathsf{sw}_{\sigma,\delta},\Delta =_{df} (\ldots(\mathsf{sw}_{\sigma,\delta},\epsilon_1)\ldots),\epsilon_n : \Gamma, x : A\ \sigma,\ y : B\ (\mathsf{wk}_\sigma\cdot\delta),\Delta \Rightarrow\\ \Gamma, y : B\ \delta, x : A\ (\mathsf{wk}_\delta\cdot\sigma), \mathsf{sw}_{\sigma,\delta}\Delta\end{aligned}$$

is a derivable substitution that is also a derivable isomorphism, in the sense that there is a derivable $(\mathsf{sw}_{\sigma,\delta},\Delta)^{-1}$ *such that*

$$\Big((\mathsf{sw}_{\sigma,\delta},\Delta)^{-1}\Big)\cdot\Big((\mathsf{sw}_{\sigma,\delta},\Delta)\Big) \equiv \mathsf{id}_{\Gamma,y:B\ \delta,x:A\ (\mathsf{wk}_\delta\cdot\sigma),\mathsf{sw}_{\sigma,\delta}\Delta}$$
$$\Big((\mathsf{sw}_{\sigma,\delta},\Delta)\Big)\cdot\Big((\mathsf{sw}_{\sigma,\delta},\Delta)^{-1}\Big) \equiv \mathsf{id}_{\Gamma,x:A\ \sigma,\ y:B\ (\mathsf{wk}_\sigma\cdot\delta),\Delta}$$

Proof. By induction on the length of Δ (as a raw expression). If $\Delta = \varnothing$ then the hypothesis follows immediately by (Sub-sw) and (swsw). Now assume that $\Delta = \Theta,\epsilon$ for some derivable

$$\epsilon : \Gamma, x : A\ \sigma,\ y : B\ (\mathsf{wk}_\sigma\cdot\delta),\Theta \Rightarrow \Sigma$$

and that the hypothesis holds for Θ. By the inductive hypothesis (IH) we are given a derivable substitution

$$\mathsf{sw}_{\sigma,\delta},\Theta : \Gamma, y : B\ \delta, x : A\ (\mathsf{wk}_\delta\cdot\sigma), \mathsf{sw}_{\sigma,\delta}\Theta \Rightarrow \Gamma, x : A\ \sigma,\ y : B\ (\mathsf{wk}_\sigma\cdot\delta),\Theta$$

and so we can define

$$\mathsf{sw}_{\sigma,\delta},\Delta =_{\text{df}} (\mathsf{sw}_{\sigma,\delta},\Theta),\epsilon \tag{1}$$

which is derivable by an application of Sub-ext. To see that $\mathsf{sw}_{\sigma,\delta},\Delta$ is an isomorphism, we set

$$(\mathsf{sw}_{\sigma,\delta},\Delta)^{-1} =_{\text{df}} (\mathsf{sw}_{\sigma,\delta},\Theta)^{-1}, ((\mathsf{sw}_{\sigma,\delta},\Theta)\cdot\epsilon) \tag{2}$$

which can easily seen to be derivable given our data. It remains to show that $(\mathsf{sw}_{\sigma,\delta},\Delta)^{-1}$ is indeed an inverse of $(\mathsf{sw}_{\sigma,\delta},\Delta)$. In one direction we have:

$$\begin{aligned}
&(\text{extcomp}) && \Big((\mathsf{sw}_{\sigma,\delta},\Theta)^{-1}, ((\mathsf{sw}_{\sigma,\delta},\Theta)\cdot\epsilon)\Big)\cdot(\mathsf{sw}_{\sigma,\delta},\Theta),\epsilon \equiv ((\mathsf{sw}_{\sigma,\delta},\Theta)^{-1}\cdot(\mathsf{sw}_{\sigma,\delta},\Theta)),\epsilon\\
&(\text{IH,swsw}) && \equiv \mathsf{id}_{\Gamma,\sigma,\mathsf{wk}_\sigma\cdot\delta,\Theta},\epsilon\\
&(\text{idext}) && \equiv \mathsf{id}_{\Gamma,\sigma,\mathsf{wk}_\sigma\cdot\delta,\Theta,\epsilon}
\end{aligned}$$

The other direction is similar, and left to the reader. □

Lemma 4.5. *Let $\Psi \vdash$, $\Psi \vdash \Gamma$ and $\Psi \vdash \Delta$ be derivable and let $\alpha : [\![\Gamma]\!] \to [\![\Delta]\!]$ be any natural transformation. Then there is a derivable substitution $\Psi \vdash \sigma : \Delta \Rightarrow \Gamma$ such that $[\![\sigma]\!] = \alpha$.*

Proof. We proceed by induction on Ψ and Γ. If $l(\Psi) = 0$ then $[\![\Psi]\!] = \varnothing$ and hence $[\![\Gamma]\!] = \varnothing$ and therefore for any $[\![\Delta]\!]$ and any $\alpha : \varnothing \to [\![\Delta]\!]$ we have $\alpha = 1_{[\![\Delta]\!]} = [\![\epsilon_\Delta]\!]$. Now assume that the hypothesis holds for all derivable Ψ with $l([\![\Psi]\!]) < n$ and let $l([\![\Psi]\!]) = n$. Take any derivable context $\Psi \vdash \Gamma$. If $h([\![\Gamma]\!]) = 0$ then $[\![\Gamma]\!] = \varnothing$ and once again the hypothesis follows. So now assume that the hypothesis holds for all derivable $\Psi \vdash \Theta$ with $h([\![\Theta]\!]) < n$ and let $h([\![\Gamma]\!]) = n$. Let $[\![\Delta]\!]$ be any (derivable) context and $\alpha : [\![\Gamma]\!] \to [\![\Delta]\!]$ any natural transformation. We need to show that this α is definable, i.e. that there is a derivable $\Psi \vdash \sigma : \Delta \Rightarrow \Gamma$ such that $\alpha = [\![\sigma]\!]$.

Now, since we assume that $[\![\Gamma]\!] \neq \varnothing$ let $A_1, \ldots, A_m \in [\![\Psi]\!]$ be the objects of height n such that $[\![\Gamma]\!](A_i) \neq \varnothing$ for $i = 1, \ldots, m$. Since α is assumed to exist, this means that $[\![\Delta]\!](A_i) \neq \varnothing$ for all $i = 1, \ldots, m$, but there are possibly some more objects $B_1, \ldots, B_k$ of height n in $[\![\Psi]\!]$ such that $[\![\Delta]\!](B_i) \neq \varnothing$. We can now write

$$[\![\Gamma]\!] = [\![\widehat{\Gamma}]\!] *_{x_1} [\![\tau_1]\!] *_{x_2} \ldots *_{x_n} [\![\tau_\nu]\!] \tag{3}$$

and

$$[\![\Delta]\!] = [\![\widehat{\Delta}]\!] *_{y_1} [\![\delta_1]\!] *_{y_2} \ldots *_{y_\mu} [\![\delta_\mu]\!] \tag{4}$$

where $[\![\widehat{\Gamma}]\!]$ is a context of height $n-1$ and $[\![\tau_j]\!]$ are the natural transformations

$$\partial x_j : \partial A_i \to [\![\widehat{\Gamma}]\!] *_{x_1} [\![\tau_1]\!] *_{x_2} \ldots *_{x_{j-1}} [\![\tau_{j-1}]\!] \tag{5}$$

for each variable $x_j \in [\![\Gamma]\!](A_i)$ for all $i = 1, \ldots, m$, and similarly for $[\![\delta_j]\!]$ (but $\widehat{[\![\Delta]\!]}$ need not be of height $n-1$ since it may be the case that $[\![\widehat{\Delta}]\!](B_i) \neq \varnothing$ for some $i = 1, \ldots, k$). By successive applications of the derivable isomorphisms described in Lemma 4.4 we may assume without loss of generality that $\alpha(x_1) = y_1$ and that if $\alpha(x_j) = y_k$ then either $\alpha(x_{j+1}) = y_k$ or $\alpha(x_{j+1}) = y_{k+1}$. It is convenient for what follows to write $[\![\tau_j^i]\!]$ if $[\![\tau_j]\!] = \partial x_j$ for some $x_j \in [\![\Gamma]\!](A_i)$, and similarly for $[\![\delta_j^i]\!]$.

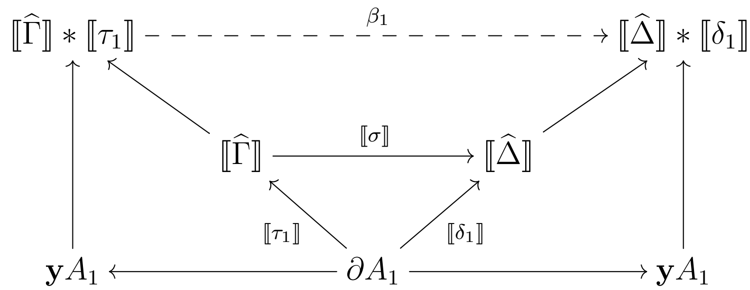
We will now proceed to define a sequence of derivable natural transformations $\beta_i : [\![\widetilde{\Gamma}]\!] \to [\![\widetilde{\Delta}]\!]$ such that $[\![\widetilde{\Gamma}]\!]$ and $[\![\widetilde{\Delta}]\!]$ are subcontexts of $[\![\Gamma]\!]$ and $[\![\Delta]\!]$ respectively, $\beta_i = [\![\sigma_i]\!]$ for some derivable σ_i and $\alpha|_{[\![\widetilde{\Gamma}]\!]} = \beta_i$. Let us write

$$\alpha|_{\widehat{\Gamma}} : \widehat{\Gamma} \to \widehat{\Delta} \tag{6}$$

for the obvious restriction of α. Since $\alpha|_{\widehat{\Gamma}}$ is a natural transformation, by the inductive hypothesis on Γ we know that there is a derivable $\sigma : \widehat{\Delta} \Rightarrow \widehat{\Gamma}$ such that

$$[\![\sigma]\!] = \alpha|_{[\![\Gamma]\!]} \tag{7}$$

With this in mind, β_1 is defined as the unique map in the following diagram



By naturality of α we know that the bottom triangle commutes, i.e. that $[\![\delta_1]\!] = [\![\sigma]\!] \circ [\![\tau_1]\!]$. But this means that

$$\beta_1 = [\![\sigma, \tau_1]\!] \tag{8}$$

and therefore β_1 is definable. It also clearly follows that $\beta_1 = \alpha|_{[\![\widehat{\Gamma}]\!]*[\![\tau_1]\!]}$, by our assumption that $\alpha(x_1) = y_1$.

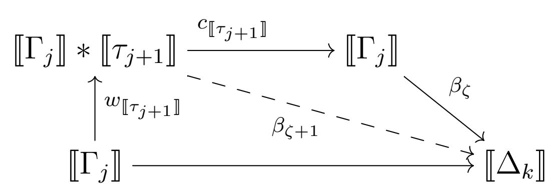
Now assume we have defined

$$\beta_\zeta : [\![\widehat{\Gamma}]\!] * [\![\tau_1]\!] * \cdots * [\![\tau^i_j]\!] \to [\![\widehat{\Delta}]\!] * [\![\delta_1]\!] * \cdots * [\![\delta^i_k]\!] \tag{9}$$

and let us write simply $[\![\Gamma_j]\!]$ for the domain β_ζ and $[\![\Delta_k]\!]$ for the codomain β_ζ. If $[\![\Gamma_j]\!] = [\![\Gamma]\!]$ and $[\![\Delta_k]\!] = [\![\Delta]\!]$ we are done. If not, there are three cases.

Case 1: τ^i_{j+1} and δ^{i+1}_{k+1}. Then $\beta_{\zeta+1}$ is defined just as above in the case of β_1 and satisfies the required properties for the same reasons.

Case 2: τ^i_{j+1}. Then we define $\beta_{\zeta+1}$ by the following diagram



To see that $\beta_{\zeta+1}$ has the required properties note that

$$\beta_{\zeta+1} =_{\mathrm{df}} \beta_\zeta \circ c_{[\![\tau_{j+1}]\!]} = [\![\sigma_\zeta \cdot \tau_{j+1}]\!] \tag{10}$$

by the assumption on β_z, and since by our re-ordering on α we know that

$$\alpha_{A_i}(x_{j+1}) = \alpha_{A_i}(x_j) = y_k \tag{11}$$

it follows precisely that

$$\beta_{\zeta+1} = \alpha|_{[\![\Gamma_j]\!]} \circ c_{[\![\tau_{j+1}]\!]} = \alpha|_{[\![\Gamma_j]\!]*[\![\tau_{j+1}]\!]} \tag{12}$$

as required.

Case 3: $j = \mu$ but $k < \nu$. This means that $[\![\Gamma_j]\!] = [\![\Gamma]\!]$, and we can define $\beta_{\zeta+1} = w_{[\![\delta_{j+1}]\!]} \circ \beta_\zeta$ as in the following diagram:

$$\begin{array}{ccc} & & [\![\Delta_j]\!] * [\![\delta_{j+1}]\!] \\ & \overset{\beta_{\zeta+1}}{\nearrow} & \Big\uparrow {\scriptstyle w_{[\![\delta_{j+1}]\!]}} \\ [\![\Gamma]\!] & \xrightarrow{\quad\beta_\zeta\quad} & [\![\Delta_j]\!] \end{array}$$

It is obvious, just as in the previous case, that $\beta_{\zeta+1}$ satisfies the required conditions.

We continue this process until we get $\beta_l : [\![\Gamma]\!] \to [\![\Delta]\!]$, in which case we have

$$\alpha = \beta_l = [\![\sigma_l]\!] \tag{13}$$

Thus α is definable, and the proof is complete. □

Lemma 4.6. *Let $\Psi \vdash$ be derivable and let G be any $[\![\Psi]\!]$-context. Then $G \cong [\![\Gamma]\!]$ for some derivable $\Psi \vdash \Gamma$.*

Proof. By induction on the height of G. If $h(G) = 0$ then $G = \varnothing = [\![\bullet]\!]$. Now assume that the hypothesis holds for contexts of height less than n and let $h(G) = n$. We can write G as the following series of pushout squares

$$\begin{array}{ccccccccc} & & \partial A_2 & \longrightarrow & \mathbf{y}A_2 & & \dots & & \mathbf{y}A_{m-2} & & \\ & & \Big\downarrow {\scriptstyle \partial x_2} & & \Big\downarrow & & & & \Big\downarrow & & \\ X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & \dots & \longrightarrow & X_{m-1} & \longrightarrow & X_m \cong G \\ {\scriptstyle \partial x_1} \Big\uparrow & & \Big\uparrow & & & & & & {\scriptstyle \partial x_{m-1}} \Big\uparrow & & \Big\uparrow \\ \partial A_1 & \longrightarrow & \mathbf{y}A_1 & & \dots & & & & \partial A_{m-1} & \longrightarrow & \mathbf{y}A_{m-1} \end{array}$$

where the x_j are all the elements of $G(A_j)$ for sorts A_j of height n and X_1 is the maximal subcontext of G of height $n-1$ (i.e. the context in which all the x_j have been removed). Then by the inductive hypothesis we know that $X_1 \cong [\![\Gamma]\!]$ and also that $\partial A_1 \cong [\![\Theta]\!]$ (since $h(\partial A_1) < n$ because the boundary of an object of height n is a context of height less than n). By Lemma 4.5 this means that $\partial x_1 = [\![\sigma_1]\!]$. Therefore,

$$X_1 = [\![\Gamma]\!] *_{x_1} [\![\sigma_1]\!] = [\![\Gamma,\, x_1 : A_1\, \sigma_1]\!] \tag{14}$$

and by the same reasoning as above we also get that $\partial x_2 = [\![\sigma_2]\!]$ and therefore $X_2 = [\![\Gamma,\, x_1 : A_1\, \sigma_1, x_2 : A_2\, \sigma_2]\!]$, and so on until we get that $G \cong X_m = [\![\Gamma, x_1 : A_1\, \sigma_1, \dots, x_m : A_m\, \sigma_m]\!]$, as was to be shown. □

Theorem 4.7. *Let $\mathcal{L}$ be a fic. Then $\mathcal{L} \cong [\![\Psi]\!]$ for some $\mathbb{T}_{fic}$-signature Ψ.*

Proof. By induction on the length of $\mathcal{L}$. If $l(\mathcal{L}) = 0$ then $\mathcal{L} = \varnothing$ and therefore $\mathcal{L} = [\![\bullet]\!]$ by definition. Now assume the hypothesis holds for all $\mathcal{L}$ with $l(\mathcal{L}) < n$. For any $K \in \operatorname{ob}\mathcal{L}$ of maximal height we have $\mathcal{L} \cong \mathcal{L}_K *_K \partial K$, where $\mathcal{L}_K$ is the category with $\operatorname{ob}\mathcal{L}_K = \operatorname{ob}\mathcal{L} \setminus \{K\}$ and $\operatorname{mor}\mathcal{L}_K = \operatorname{mor}\mathcal{L} \setminus \{f \mid \operatorname{dom}(f) = K\}$. ($\mathcal{L}_K$ is well-defined because we know that K is a sort of maximal height in $\mathcal{L}$.) Therefore, it suffices to show $\partial K = [\![\Gamma]\!]$ for some $\Gamma \in \mathbf{Con}([\![\Psi]\!])$ where $[\![\Psi]\!] = \mathcal{L}_K$ (using the inductive hypothesis to obtain Ψ). But this is exactly Lemma 4.6. □

For some future applications (e.g. [WT]) it is important to state Theorem 4.7 as a bijection between (an appropriate notion) of the set of all fics and the set of all derivable signatures in the sense of $\mathbb{T}_{\text{fic}}$.

Definition 4.8. For any fic $\mathcal{L}$ a **proper ordering** $<$ on $\mathcal{L}$ is a pair $(<_o, <_m)$ where $<_o$ is a partial order on $\text{ob}\,\mathcal{L}$ such that $\exists f\colon K' \to K \Rightarrow K <_o K'$, and for any $K \in \text{ob}\,\mathcal{L}$ a partial order $<_m$ in $K \downarrow \mathcal{L}$ such that $h = g \circ f \Rightarrow f <_m h$. We write $\mathbf{FIC}_<$ for the set of pairs $(\mathcal{L}, <)$ where $\mathcal{L}$ is a fic and $(<_o, <_m)$ is a proper ordering on $\mathcal{L}$.

The interpretation function $[\![-]\!]$ given in Definition 3.7 extends in an obvious way to an interpretation function $[\![-]\!]_<$ with values in $\mathbf{FIC}_<$ by extracting a proper ordering from the order in which contexts and sorts appear in a signature expression. Let us write *Sig* for the set of well-formed signatures, i.e. of all Ψ such that $\mathbb{T}_{\text{fic}} \models \Psi$. Then we have:

Corollary 4.9. $[\![-]\!]_<$ *defines a bijection* $\boldsymbol{FIC}_< \simeq Sig$*.*

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