Category Theory Pset 2

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1. Let Σ be the one-sorted first-order signature consisting of a function symbol $f\colon S \to S$. Show that every Σ-structure $\mathcal{M}$ can be expressed as a functor $M_{\mathcal{M}}$ in the category $\mathbf{Set}^D$ for some appropriately chosen D. Conversely, show that every functor M in $\mathbf{Set}^D$ gives rise to a Σ-structure $\mathcal{M}_M$. Hence, prove that there is an equivalence between the category $\mathbf{Str}(\Sigma)$ with objects the Σ-structures and Σ-homomorphisms as arrows and the functor category $\mathbf{Set}^D$ (for your appropriately chosen D).

2. *For R a ring, describe the category R-$\mathbf{Mod}$ of R-modules and R-module homomorphisms as a full subcategory of the functor category $\mathbf{Ab}^R$.

3. For $\mathcal{P}$ and $\mathcal{Q}$ posets, describe the functor category $\mathcal{Q}^{\mathcal{P}}$ and prove that it is a poset.

4. Let $\mathcal{C}$ be a category and c an object in $\mathcal{C}$.

   (a) Prove that the category $(\mathcal{C}/c)^{op}$ is equivalent to the category of elements of the functor $\mathcal{C}(-, c)$. (read definition 2.4.1)

   (b) Prove that the co-slice category $c/\mathcal{C}$ is equivalent to the category of elements of the functor $\mathcal{C}(c, -)$.

5. Given functors $T, S\colon \mathcal{D} \to \mathcal{C}$ show that a natural transformation $\tau\colon T \Rightarrow S$ is the same thing as a functor $\tau\colon \mathcal{D} \to (T \downarrow S)$ such that $P\tau = Q\tau = 1_{\mathcal{D}}$, with $P\colon T \downarrow S \to \mathcal{D}$ the projection functor that on objects sends $(a, b, f) \mapsto a$ and $Q\colon T \downarrow S \to \mathcal{D}$ the projection functor that on objects sends $(a, b, f) \mapsto b$. (read definition in 1.3.vi)

6. *Show that the functor $\mathcal{O}\colon \mathbf{Top} \to \mathbf{Set}$ which sends a topological space X to its set of open subsets is represented by the Sierpinski space, i.e. by the topology on $\{0, 1\}$ where exactly one of its two proper subsets is open.

7. Define a functor $M\colon \mathbf{Cat} \to \mathbf{Set}$ that sends a (small) category to the set of all its morphisms. Hence, prove that it is representable.

8. For $F\colon \mathcal{C} \to \mathbf{Set}$ and any a in $\mathcal{C}$ prove that there is an isomorphism

$$\mathrm{Nat}(F, \mathcal{C}(a, -)) \simeq \mathrm{Nat}(\hat{a}, P)$$

natural in F and a, where $P\colon \int F \to \mathcal{C}$ is the canonical projection functor that on objects sends $(a, x) \mapsto a$ and $\hat{a}$ is the constant functor on a.