Category Theory Pset 1

September 24, 2018

1. Is there a faithful functor $F\colon \mathcal{C} \to \mathcal{D}$ such that there exist distinct arrows f, g in $\mathcal{C}$ with $F(f) = F(g)$? Provide an example, or prove that no example exists.

2. Let $\mathcal{C}$ be a category and $\mathcal{C}_g$ its associated groupoid. Define a faithful functor $\mathcal{C}_g \to \mathcal{C}$. Hence, or otherwise, prove that if $\mathcal{C}$ is equivalent to $\mathcal{C}_g$ then $\mathcal{C}$ is also a groupoid.

3. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a full and faithful functor.

   (a) Show that F is *conservative*: For any arrow $f\colon a \to b$, if Ff is an isomorphism then f is an isomorphism.

   (b) Show that F *creates isomorphisms*: For any objects a, b in $\mathcal{C}$, if $Fa \cong Fb$ then $a \cong b$.

4. Riehl, Exercise 1.1.iii (p.8)

5. Let $(\mathbb{P}, \leq)$ be a partially ordered set ("poset").

   (a) Describe a category structure on $\mathbb{P}$ such that there is an arrow between any $a, b \in \mathbb{P}$ iff $a \leq b$.

   (b) Hence, define a category **Poset** with objects posets and morphisms the order-preserving maps between posets. (Given posets $(\mathbb{P}_1, <)$ and $(\mathbb{P}_2, \leq)$ a map $f\colon \mathbb{P}_1 \to \mathbb{P}_2$ is *order preserving* iff $a < b \Rightarrow f(a) \leq f(b)$).

   (c) Use your previous construction to define a full functor **Poset** $\to$ **Cat**.

   (d) Prove that **Poset** is equivalent to the category of (small) categories that have at most one arrow between any two of their objects.

6. Find a set A such that $\mathbf{Set}(A, -)$ is naturally isomorphic to the identity functor on **Set**.

7. Let $\mathcal{C}$ be a locally small category. Prove that $f\colon a \to b$ is an isomorphism if and only if for any c in $\mathcal{C}$ the "precomposition" function

$$\mathcal{C}(b, c) \longrightarrow \mathcal{C}(a, c)$$

$$g \longmapsto g \circ f$$

   is a bijection.